

Chapter 5

Compton scattering

5.1 Introduction

The simplest interaction between photons and free electrons is *scattering*. When the energy of the incoming photons (as seen in the comoving frame of the electron) is small with respect to the electron rest mass-energy, the process is called *Thomson scattering*, which can be described in terms of classical electro-dynamics. As the energy of the incoming photons increases and becomes comparable or greater than $m_e c^2$, a quantum treatment is necessary (Klein–Nishina regime).

5.2 The Thomson cross section

Assume an electron at rest, and an electromagnetic wave of frequency $\nu \ll m_e c^2/h$. Assume also that the incoming wave is completely linearly polarized. In order to neglect the magnetic force $(e/c)(\mathbf{v} \times \mathbf{B})$ we must also require that the oscillation velocity $v \ll c$. This in turn implies that the incoming wave has a sufficiently low amplitude. The electron starts to oscillate in response to the varying electric force eE , and the average square acceleration during one cycle of duration $T = 1/\nu$ is

$$\langle a^2 \rangle = \frac{1}{T} \int_0^T \frac{e^2 E_0^2}{m_e^2} \sin^2(2\pi\nu t) dt = \frac{e^2 E_0^2}{2m_e^2} \quad (5.1)$$

The emitted power per unit solid angle is given by the Larmor formula $dP/d\Omega = e^2 a^2 \sin^2 \Theta / (4\pi c^3)$ where Θ is the angle between the acceleration vector and the propagation vector of the emitted radiation. Please note that Θ is *not* the scattering angle, which is instead the angle between the incoming and the scattered wave (or photon). We then have

$$\frac{dP_e}{d\Omega} = \frac{e^4 E_0^2}{8\pi m_e^2 c^3} \sin^2 \Theta \quad (5.2)$$

The scattered radiation is completely linearly polarized in the plane defined by the incident polarization vector and the scattering direction. The flux of the incoming wave is $S_i = cE_0^2/(8\pi)$. The differential cross section of the process is then

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{pol}} = \frac{dP_e/d\Omega}{S_i} = r_0^2 \sin^2 \Theta \quad (5.3)$$

where $r_0 \equiv e^2/(m_e c^2)$ is the *classic electron radius*, $r_0 = 2.82 \times 10^{-13}$ cm. We see that the scattered pattern of a completely polarized incoming wave is a torus, with axis along the acceleration direction. The total cross section can be derived in a similar way, but considering the Larmor formula integrated over the solid angle [$P = 2e^2 a^2/(3c^3)$]. In this way the total cross section is

$$\sigma_{\text{pol}} = \frac{P_e}{S_i} = \frac{8\pi}{3} r_0^2 \quad (5.4)$$

Note that the classical electron radius can also be derived by equating the energy of the associated electric field to the electron rest mass-energy:

$$m_e c^2 = \int_{a_0}^{\infty} \frac{E^2}{8\pi} 4\pi r^2 dr = \int_{a_0}^{\infty} \frac{e^2}{2r^2} dr \rightarrow a_0 = \frac{1}{2} \frac{e^2}{m_e c^2} \quad (5.5)$$

Why is a_0 slightly different from r_0 ? Because there is an intrinsic uncertainty related to the distribution of the charge within (or throughout the surface of) the electron. See the discussion in Vol. 2, chapter 28.3 of “The Feynman Lectures on Physics”, about the fascinating idea that the mass of the electron is all electromagnetic.

5.2.1 Why the peanut shape?

The scattering of a completely unpolarized incoming wave can be derived by assuming that the incoming radiation is the sum of two orthogonal completely linearly polarized waves, and then summing the associated scattering patterns. Since we have the freedom to chose the orientations of the two polarization planes, it is convenient to chose one of these planes as the one defined by the incident and scattered directions, and the other one perpendicular to this plane. The scattering can be then regarded as the sum of two independent scattering processes, one with emission angle Θ , the other with $\pi/2$. If we note that the scattering angle (i.e. the angle between the scattered wave and the incident wave) is $\theta = \pi/2 - \Theta$, we have

$$\begin{aligned} \left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}} &= \frac{1}{2} \left[\left(\frac{d\sigma(\Theta)}{d\Omega}\right)_{\text{pol}} + \left(\frac{d\sigma(\pi/2)}{d\Omega}\right)_{\text{pol}} \right] \\ &= \frac{1}{2} r_0^2 (1 + \sin^2 \Theta) \\ &= \frac{1}{2} r_0^2 (1 + \cos^2 \theta) \end{aligned} \quad (5.6)$$

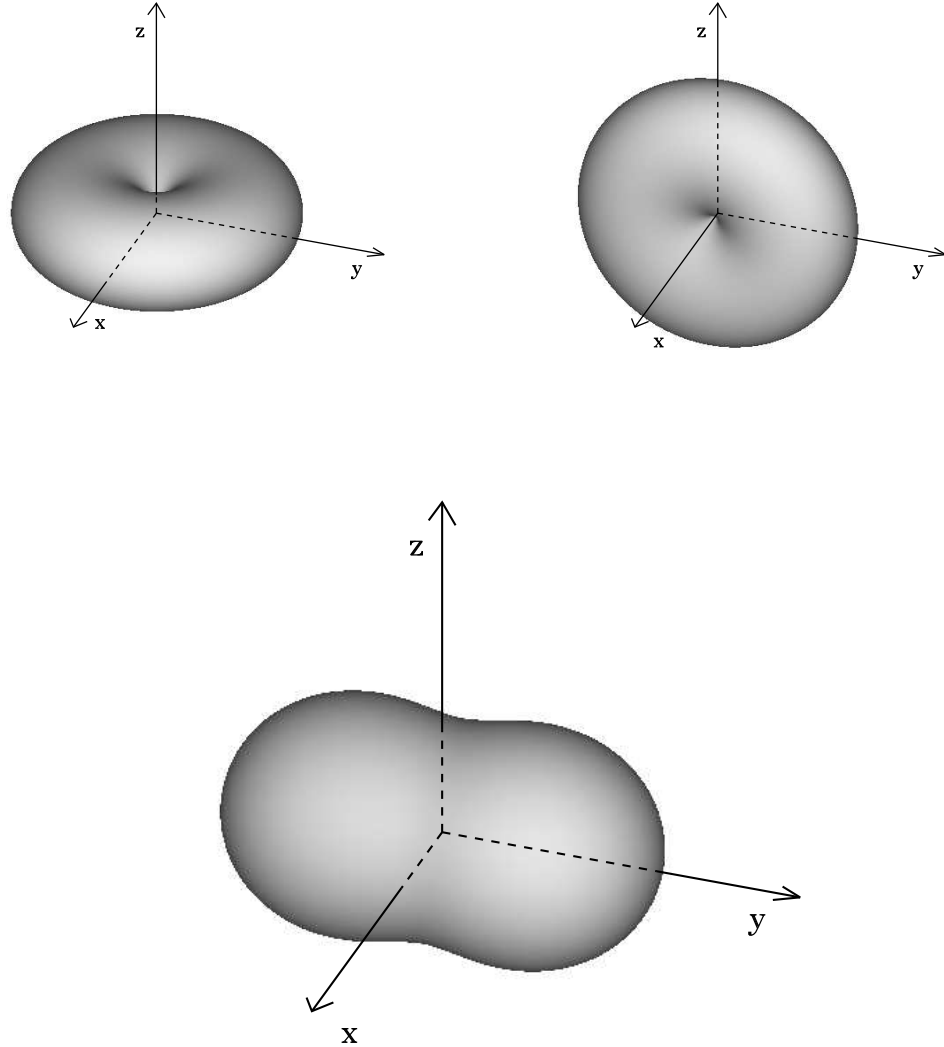


Figure 5.1: Photons are coming along the y -axis. The top panels shows the pattern of the scattered radiation for photons completely linearly polarized along the z -axis (left) and along the x -axis (right). The sum of the two torii corresponds to the pattern for unpolarized radiation (bottom panel). This explains why we have a “peanut” shape, elongated along the velocity vector of the incoming photons. Courtesy of Davide Lazzati.

In this case we see that the cross section depends only on the scattering angle θ . The pattern of the scattered radiation is then the superposition of two orthogonal “tori” (one for each polarization direction), as illustrated in Fig. 5.1. When scattering completely linearly polarized radiation, only one “torus” survives. Instead, when scattering unpolarized radiation, some polarization is introduced, because of the difference between the two “tori” patterns. Both terms of the RHS of Eq. 5.6 refer to completely polarized scattered waves (but in two perpendicular planes). The difference between these two terms is then associated to the introduced polarization, which is then

$$\Pi = \frac{1 - \cos^2 \theta}{1 + \cos^2 \theta} \quad (5.7)$$

The above discussion helps to understand why the scattering process introduces some polarization, which is maximum (100%) if the angle between the incoming and the scattered photons is 90° (only one torus contributes), and zero for 0° and 180° , where the two torii give the same contribution.

The total cross section, integrated over the solid angle, is the same as that for polarized incident radiation (Eq. 5.4) since the electron at rest has no preferred defined direction. This is the Thomson cross section:

$$\begin{aligned} \sigma_T &= \int \left(\frac{d\sigma}{d\Omega} \right)_{\text{unpol}} d\Omega = \frac{2\pi r_0^2}{2} \int (1 + \cos^2 \theta) d\cos \theta = \frac{8\pi}{3} r_0^2 \\ &= 6.65 \times 10^{-25} \text{ cm}^2 \end{aligned} \quad (5.8)$$

5.3 Direct Compton scattering

In the previous section we considered the scattering process as an interaction between an electron and an *electromagnetic wave*. This required $h\nu \ll m_e c^2$. In the general case the quantum nature of the radiation must be taken into account. We consider then the scattering process as a collision between the electron and the *photon*, and apply the conservation of energy and momentum to derive the energy of the scattered photon. It is convenient to measure energies in units of $m_e c^2$ and momenta in units of $m_e c$.

Consider an electron at rest and an incoming photon of energy x_0 , which becomes x_1 after scattering. Let θ be the angle between the incoming and outgoing photon directions. This defines the scattering plane. Momentum conservation dictates that also the momentum vector of the electron, after the scattering, lies in the same plane. Conservation of energy and conservation of momentum along the x and y axis gives:

$$x_1 = \frac{x_0}{1 + x_0(1 - \cos \theta)} \quad (5.9)$$

Note that, for $x_0 \gg 1$ and $\cos \theta \neq 1$, $x_1 \rightarrow (1 - \cos \theta)^{-1}$. In this case the scattered photon carries information about the scattering angle, rather than

about the initial energy. As an example, for $\theta = \pi$ and $x_0 \gg 1$, the final energy is $x_1 = 0.5$ (corresponding to 255 keV) independently of the exact value of the initial photon energy. Note that for $x_0 \ll 1$ the scattered energy $x_1 \simeq x_0$, as assumed in the classical Thomson scattering. The energy shift implied by Eq. 5.9 is due to the recoil of the electron originally at rest, and becomes significant only when x_0 becomes comparable with 1 (or more). When the energy of the incoming photon is comparable to the electron rest mass, another quantum effect appears, namely the energy dependence of the cross section.

5.4 The Klein–Nishina cross section

The Thomson cross section is the classical limit of the more general Klein–Nishina cross section (here we use x as the initial photon energy, instead of x_0 , for simplicity):

$$\frac{d\sigma_{\text{KN}}}{d\Omega} = \frac{3}{16\pi} \sigma_{\text{T}} \left(\frac{x_1}{x} \right)^2 \left(\frac{x}{x_1} + \frac{x_1}{x} - \sin^2 \theta \right) \quad (5.10)$$

This is a compact form, but there appears dependent quantities, as $\sin \theta$ is related to x and x_1 . By inserting Eq. 5.9, we arrive to

$$\frac{d\sigma_{\text{KN}}}{d\Omega} = \frac{3}{16\pi} \frac{\sigma_{\text{T}}}{[1 + x(1 - \cos \theta)]^2} \left[x(1 - \cos \theta) + \frac{1}{1 + x(1 - \cos \theta)} + \cos^2 \theta \right] \quad (5.11)$$

In this form, only independent quantities appear (i.e. there is no x_1). Note that the cross section becomes smaller for increasing x and that it coincides with $d\sigma_{\text{T}}/d\Omega$ for $\theta = 0$ (for this angle $x_1 = x$ independently of x). This however corresponds to a vanishingly small number of interactions, since $d\Omega \rightarrow 0$ for $\theta \rightarrow 0$).

Integrating Eq. 5.11 over the solid angle, we obtain the total Klein–Nishina cross section:

$$\sigma_{\text{KN}} = \frac{3}{4} \sigma_{\text{T}} \left\{ \frac{1+x}{x^3} \left[\frac{2x(1+x)}{1+2x} - \ln(1+2x) \right] + \frac{1}{2x} \ln(1+2x) - \frac{1+3x}{(1+2x)^2} \right\} \quad (5.12)$$

Asymptotic limits are:

$$\begin{aligned} \sigma_{\text{KN}} &\simeq \sigma_{\text{T}} \left(1 - 2x + \frac{26x^2}{5} + \dots \right); & x \ll 1 \\ \sigma_{\text{KN}} &\simeq \frac{3}{8} \frac{\sigma_{\text{T}}}{x} \left[\ln(2x) + \frac{1}{2} \right]; & x \gg 1 \end{aligned} \quad (5.13)$$

The direct Compton process implies a transfer of energy from the photons to the electrons. It can then be thought as an heating mechanism. In the next

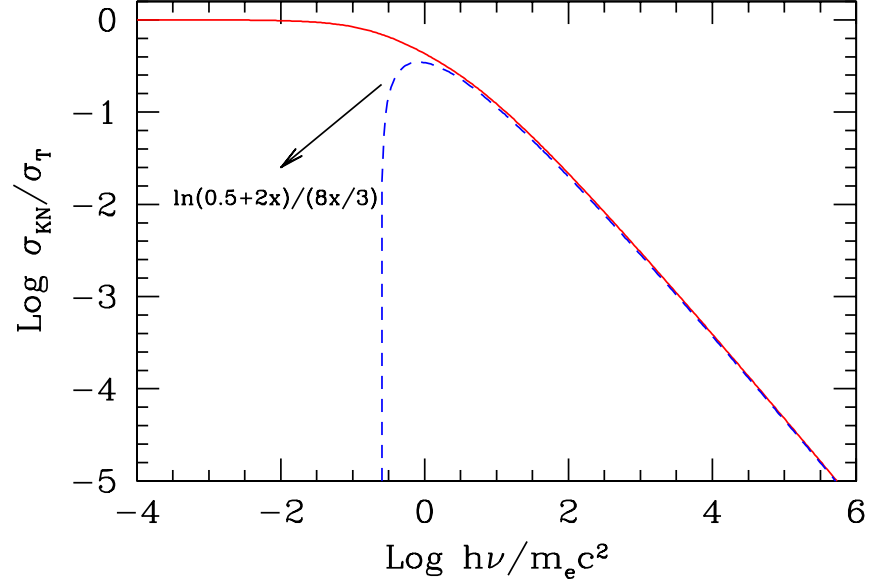


Figure 5.2: The total Klein–Nishina cross section as a function of energy. The dashed line is the approximation at high energies as given in Eq. 5.13.

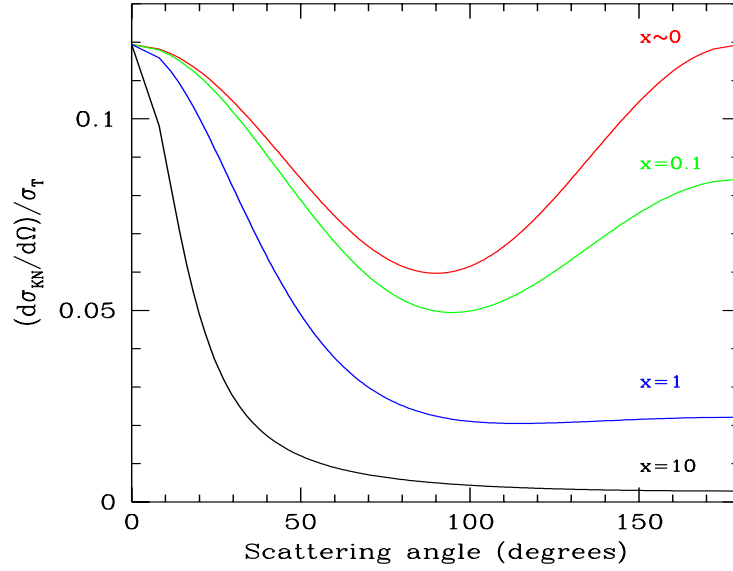


Figure 5.3: The differential Klein–Nishina cross section (in units of σ_T), for different incoming photon energies. Note how the scattering becomes preferentially forward as the energy of the photon increases.

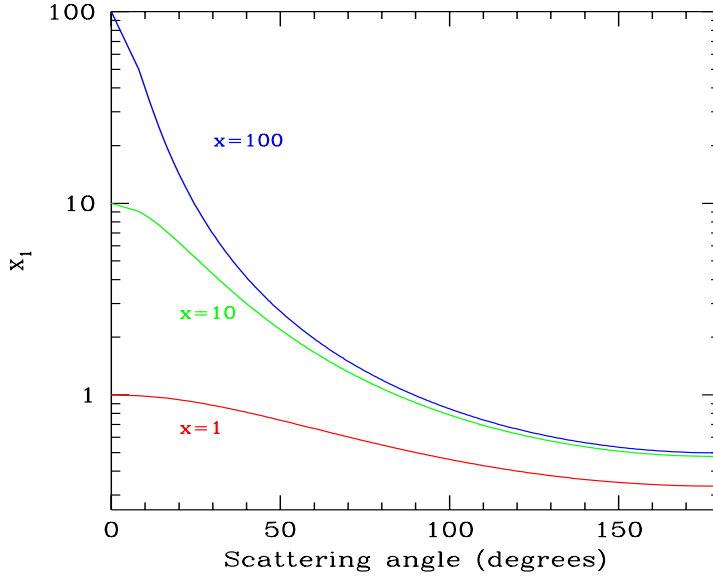


Figure 5.4: Scattered photons energies as a function of the scattering angle, for different incoming photon energies. Note that, for $x \gg 1$ and for large scattering angle, the scattered photon energies becomes $x_1 \sim 1/2$, independent of the initial photon energy x .

subsection we discuss the opposite process, called *inverse Compton scattering*, in which hot electrons can transfer energy to low frequency photons.

We have so far neglected the momentum exchange between radiation and the electron. One can see, even classically, that there must be a net force acting along the direction of the wave if one considers the action of the magnetic field of the wave. In fact the Lorentz force $e\mathbf{v} \times \mathbf{B}$ is *always* directed along the direction of the wave (here $\mathbf{v}$ is the velocity along the $\mathbf{E}$ field). This explains the fact that light can exert a pressure, even classically.

5.4.1 Another limit

We have mentioned that, in order for the magnetic Lorentz force to be negligible, the electron must have a transverse (perpendicular to the incoming wave direction) velocity $\ll c$. Considering a wave of frequency ω and electric field $E = E_0 \sin(\omega t)$, this implies that:

$$\frac{v_{\perp}}{c} = \int_0^{T/2} \frac{eE_0}{cm_e} \sin(\omega t) dt = \frac{2eE_0}{m_e c \omega} \ll 1 \quad (5.14)$$

This means that the scattering process can be described by the Thomson cross section if the wave have a sufficiently low amplitude and a not too small frequency (i.e. for very small frequencies the electric field of the wave accelerates the electron for a long time, and then to large velocities).

5.4.2 Pause

Now pause, and ask if there are some ways to apply what we have done up to now to real astrophysical objects.

- The Eddington luminosity is derived with the Thomson cross section, with the thought that it describes the smallest probability of interaction between matter and radiation. But the Klein–Nishina cross section can be even smaller, as long as the source of radiation emits at high energy. What are the consequences? If you have forgotten the definition of the Eddington luminosity, here it is:

$$L_{\text{Edd}} = \frac{4\pi GMm_{\text{p}}c}{\sigma_{\text{T}}} = 1.3 \times 10^{38} \frac{M}{M_{\odot}} \text{ erg s}^{-1} \quad (5.15)$$

- In Nova Muscae, some years ago a (transient) annihilation line was detected, together with another feature (line-like) at 200 keV. What can this feature be?
- It seems that high energy radiation can suffer less scattering and therefore can propagate more freely through the universe. Is that true? Can you think of other processes that can kill high energy photons in space?
- Suppose to have an astrophysical source of radiation very powerful above say – 100 MeV. Assume that at some distance there is a very efficient “reflector” (i.e. free electrons) and that you can see the scattered radiation. Can you guess the spectrum you receive? Does it contain some sort of “pile-up” or not? Will this depend upon the scattering angle?

5.5 Inverse Compton scattering

When the electron is not at rest, but has an energy greater than the typical photon energy, there can be a transfer of energy from the electron to the photon. This process is called *inverse* Compton to distinguish it from the *direct* Compton scattering, in which the electron is at rest, and it is the photon to give part of its energy to the electron.

We have two regimes, that are called the *Thomson* and the *Klein–Nishina* regimes. The difference between them is the following: we go in the frame where the electron is at rest, and in that frame we calculate the energy of the incoming photon. If the latter is smaller than $m_e c^2$ we are in the Thomson regime. In this case the recoil of the electron, even if it always exists, is small, and can be neglected. In the opposite case (photon energies larger than $m_e c^2$), we are in the Klein–Nishina one, and we cannot neglect the recoil. As we shall see, in both regimes the typical photon *gain* energy, even