Chapter 2

Bremsstrahlung and black body

2.1 Bremsstrahlung

We will follow an approximate derivation. For a more complete treatment see Rybicki & Lightman (1979) and Blumental & Gould (1970). We will consider an electron–proton plasma.

Definitions:

- b: impact parameter
- v: velocity of the electron
- n_{e} number density of the electrons
- n_{p}: number density of the protons
- T: temperature of the plasma: $mv^2 \sim kT \to v \sim (kT/m)^{1/2}$.

We here calculate the total power and also the spectrum of bremsstrahlung radiation. We divide the procedure into a few steps:

1. We consider the interaction between the electron and the proton only when the electron passes close to the proton. The characteristic time τ is
$$\tau \approx \frac{b}{v} \tag{2.1}$$

2. During the interaction we assume that the acceleration is constant and equal to
$$a \approx \frac{e^2}{m_e b^2} \tag{2.2}$$

3. From the Larmor formula we get

$$P = \frac{2e^2a^2}{3c^3} \approx \frac{e^2}{c^3}\frac{e^4}{m_e^2c^3b^4} = \frac{e^6}{m_e^2c^3b^4} \tag{2.3}$$

   Note that we have dropped the 2/3 factor, since in this simplified treatment we neglect all the numerical factors of order unity. Later we will give the exact result.

4. Since there is a characteristic time, there is also a characteristic frequency, namely τ^{-1}:

$$\omega \approx \frac{1}{\tau} = \frac{v}{b} \tag{2.4}$$

5. Therefore

$$P(\omega) \approx \frac{P}{\omega} = \frac{e^6}{m_e^2c^3vb^3} \tag{2.5}$$

6. We can estimate the impact factor b from the density of protons:

$$b \approx n_p^{-1/3} \to b^3 = \frac{1}{n_p} \tag{2.6}$$

7. The emissivity $j(\omega)$ will be the power emitted by a single electron multiplied by the number density of electrons. If the emission is isotropic we have also to divide by 4π, since the emissivity is for unit solid angle:

$$j(\omega) \approx \frac{n_e n_p}{4\pi}\frac{e^6}{m_e^2c^3}\left(\frac{m_e}{kT}\right)^{1/2} \tag{2.7}$$

8. We integrate $j(\omega)$ over frequency. The integral will depend upon ω_{max}. What should we use for ω_{max}? One possibility is to set $\hbar\omega_{max} = kT$. This would mean that an electron cannot emit a photon of energy larger than the typical energy of the electron. Seems reasonable, but we are forgetting all the electrons (and the frequencies) that have energies larger than kT. In this way:

$$\begin{aligned} j &= \int_0^{\omega_{max}} j(\omega)d\omega \sim \frac{n_e n_p}{4\pi}\frac{e^6}{m_e^2c^3}\left(\frac{m_e}{kT}\right)^{1/2}\frac{kT}{\hbar} \\ &= \frac{n_e n_p e^6}{4\pi m_e^2 c^3}\frac{(m_e kT)^{1/2}}{\hbar} \end{aligned} \tag{2.8}$$

   We suspect that in the exact results there will be the contribution of electrons with energy larger than kT: since they belong to the exponential part of the Maxwellian, we suspect that in the exact result there will be an exponential...

9. The exact result, considering also that $\nu = \omega/(2\pi)$, is

$$\begin{aligned} j(\nu) &= \frac{8}{3}\left(\frac{2\pi}{3}\right)^{1/2}\frac{n_\mathrm{e}n_\mathrm{p}e^6}{m_\mathrm{e}^2c^3}\left(\frac{m_\mathrm{e}}{kT}\right)^{1/2} e^{-h\nu/kT}\,\bar{g}_\mathrm{ff} \\ j &= \frac{4}{3\pi}\left(\frac{2\pi}{3}\right)^{1/2}\frac{n_\mathrm{e}n_\mathrm{p}e^6}{m_\mathrm{e}^2c^3}\frac{(m_\mathrm{e}kT)^{1/2}}{\hbar}\,\bar{g}_\mathrm{ff} \end{aligned} \tag{2.9}$$

The Gaunt factor $\bar{g}_\mathrm{ff}$ depends on the minimum impact factor which in turn determines the maximum frequency. Details are complicated, but see Rybicki & Lightman (p. 158–161) for a more detailed discussion.

We have treated the case of an electron–proton plasma. In the more general case, the plasma will be composed by nuclei with atomic number Z and number density n. The emissivity will then be proportional to Z^2. This is because the acceleration of the electron will be $a = Ze^2/(m_\mathrm{e}b^2)$ (see point 2), and we have to square the acceleration to get the power from the Larmor formula. In cgs units we have:

$$\begin{aligned} j(\nu) &= 5.4\times 10^{-39}\, Z^2 n_\mathrm{e}n_\mathrm{i}T^{-1/2}e^{-h\nu/kT}\,\bar{g} \\ j &= 1.13\times 10^{-28}\, Z^2 n_\mathrm{e}n_\mathrm{i}T^{1/2}\bar{g} \end{aligned} \tag{2.10}$$

2.1.1 Free–free absorption

If the underlying particle distribution is a Maxwellian, we can use the Kirchoff law to find out the absorption coefficient. If B_ν is the intensity of black body emission, we must have

$$S_\nu \equiv \frac{j_\nu}{\alpha_\nu} = B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1} \tag{2.11}$$

In these cases it is very simple to find α_ν once we know j_ν. Remember: this can be done only if we have a Maxwellian. If the particle distribution is non–thermal, we cannot use the Kirchoff law and we have to go back to a more fundamental level, namely to the Einstein coefficients. Using Eg. 2.11 we have:

$$\alpha_\nu^\mathrm{ff} = \frac{j_\nu}{B_\nu} = \frac{4}{3}\left(\frac{2\pi}{3}\right)^{1/2}\frac{Z^2n_\mathrm{e}n_\mathrm{i}e^6}{hm_\mathrm{e}^2c^2}\left(\frac{m_\mathrm{e}c^2}{kT}\right)^{1/2}\frac{1-e^{-h\nu/kT}}{\nu^3}\,\bar{g}_\mathrm{ff} \tag{2.12}$$

In cgs units [cm^{-1}] we have

$$\alpha_\nu^\mathrm{ff} = 3.7\times 10^8\,\frac{Z^2n_\mathrm{e}n_\mathrm{i}}{T^{1/2}}\,\frac{1-e^{-h\nu/kT}}{\nu^3}\,\bar{g}_\mathrm{ff} \tag{2.13}$$

When $h\nu \ll kT$ (Raleigh–Jeans regime) this simplifies to

$$\alpha_\nu^\mathrm{ff} = 0.018\,\frac{Z^2n_\mathrm{e}n_\mathrm{i}}{T^{3/2}\nu^2}\,\bar{g}_\mathrm{ff} \tag{2.14}$$

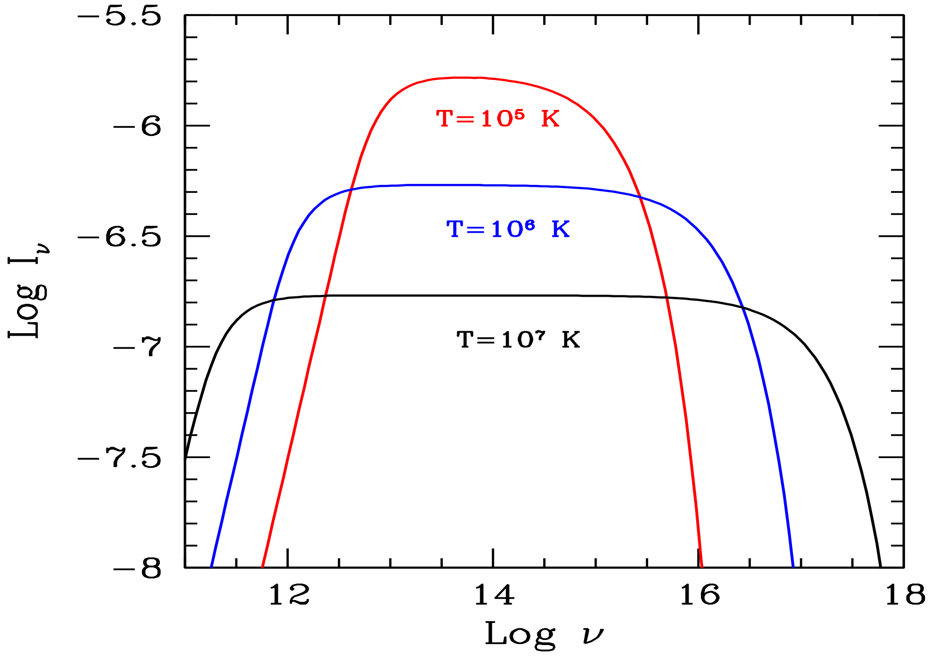

Figure 2.1: The bremsstrahlung intensity from a source of radius $R = 10^{15}$ cm, density $n_e = n_p = 10^{10}$ cm^{-3} and varying temperature. The Gaunt factor is set to unity for simplicity. At smaller temperatures the thin part of I_ν is larger ($\propto T^{-1/2}$), even if the frequency integrated I is smaller ($\propto T^{1/2}$).

Fig. 2.1 shows the bremsstrahlung intensity from a source of radius $R = 10^{15}$ cm and $n_e = n_p = 10^{10}$ cm^{-3}. The three spectra correspond to different temperatures. Note that for smaller temperatures the thin part of I_ν is larger ($I_\nu \propto T^{-1/2}$). On the other hand, at larger T the spectrum extends to larger frequencies, making the frequency integrated intensity to be larger for larger T ($I \propto T^{1/2}$). Note also the self–absorbed part, whose slope is proportional to ν^2. This part ends when the optical depth $\tau = \alpha_\nu R \sim 1$.

2.1.2 From bremsstrahlung to black body

As any other radiation process, the bremsstrahlung emission has a self–absorbed part, clearly visible in Fig. 2.1. This corresponds to optical depths $\tau_\nu \gg 1$. The term ν^{-3} in the absorption coefficient α_ν ensures that the absorption takes place preferentially at low frequencies. By increasing the density of the emitting (and absorbing) particles, the spectrum is self–absorbed up to larger and larger frequencies. When *all* the spectrum is self absorbed (i.e. $\tau_\nu > 1$ for all ν), *and* the particles belong to a Maxwellian, then we have a black–body. This is illustrated in Fig. 2.2: all spectra are calculated for the same source size ($R = 10^{15}$ cm), same temperature ($T = 10^7$ K), and what varies is the density of electrons and protons (by a factor 10) from

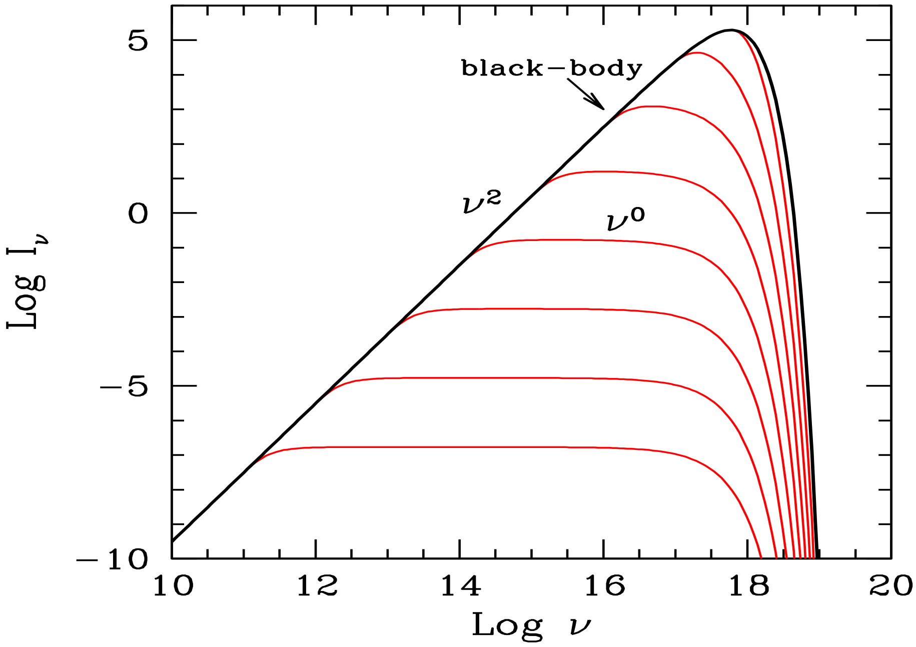

Figure 2.2: The bremsstrahlung intensity from a source of radius $R = 10^{15}$ cm, temperature $T = 10^7$ K. The Gaunt factor is set to unity for simplicity. The density $n_e = n_p$ varies from 10^{10} cm^{-3} (bottom curve) to 10^{18} cm^{-3} (top curve), increasing by a factor 10 for each curve. Note the self–absorbed part ($\propto \nu^2$), the flat and the exponential parts. As the density increases, the optical depth also increases, and the spectrum approaches the black–body one.

$n_e = n_p = 10^{10}$ cm^{-3} to 10^{18} cm^{-3}. As can be seen, the bremsstrahlung intensity becomes more and more self–absorbed as the density increases, until it becomes a black–body. At this point increasing the density does not increase the intensity any longer. This is because we receive radiation from a layer of unity optical depth. The width of this layer decreases as we increase the densities, but the emissivity increases, so that

$$I_\nu = \frac{j_\nu R}{\tau_\nu} \propto \frac{n_e n_p R}{n_e n_p R} \to \text{constant} \qquad (\tau_\nu \gg 1) \qquad (2.15)$$

2.2 Black body

A black body occurs when "the body is black": it is the perfect absorber. But this means that it is also the "perfect" emitter, since absorption and emission are linked. The black body intensity is given by

$$B_\nu(T) \;=\; \frac{2}{c^2}\,\frac{h\nu^3}{e^{h\nu/kT}-1} \tag{2.16}$$

Expressed in terms of the wavelength λ this is equivalent to:

$$B_\lambda(T) \;=\; \frac{2hc^2}{\lambda^5}\,\frac{1}{e^{hc/\lambda kT}-1} \tag{2.17}$$

Note the following:

- The black body intensity has a peak. The value of it is different if we ask for the peak of B_ν or the peak of νB_ν.
  The first is at $h\nu_{\rm peak} = 2.82\,kT$.
  The second is at $h\nu_{\rm peak} = 3.93\,kT$.
- If $T_2 > T_1$, then: $B_\nu(T_2) > B_\nu(T_1)$ for all frequencies.
- When $h\nu \ll kT$ we can expand the exponential term: $e^{h\nu/kT} \to 1 + h\nu/kT...$, and therefor we obtain the **Raleigh–Jeans law**:

$$I_\nu^{\rm RJ} \;=\; \frac{2\nu^2}{c^2}\,kT \tag{2.18}$$

- When $h\nu \gg kT$ we have $e^{h\nu/kT} - 1 \to e^{h\nu/kT}$ and we obtain the **Wien law**:

$$I_\nu^{\rm W} \;=\; \frac{2h\nu^3}{c^2}e^{-h\nu/kT} \tag{2.19}$$

- The integral over frequencies is:

$$\int_0^\infty B_\nu d\nu \;=\; \frac{\sigma_{\rm MB}}{\pi}\,T^4, \qquad \sigma_{\rm MB} \;=\; \frac{2\pi^5 k^4}{15c^2h^3} \tag{2.20}$$

  The constant $\sigma_{\rm MB}$ is called Maxwell–Boltzmann constant.

- The energy density u of black body radiation is

$$u \;=\; \frac{4\pi}{c}\int_0^\infty B_\nu d\nu \;=\; a\,T^4, \qquad a \;=\; \frac{4\sigma_{\rm MB}}{c} \tag{2.21}$$

  The two constants (σ_{MB} and a) have the values:

$$\begin{aligned} \sigma_{MB} &= 5.67\times 10^{-5} \quad {\rm erg\,cm^{-2}\,deg^{-4}\,s^{-1}} \\ a &= 7.65\times 10^{-15} \quad {\rm erg\,cm^{-3}\,deg^{-4}} \end{aligned} \tag{2.22}$$

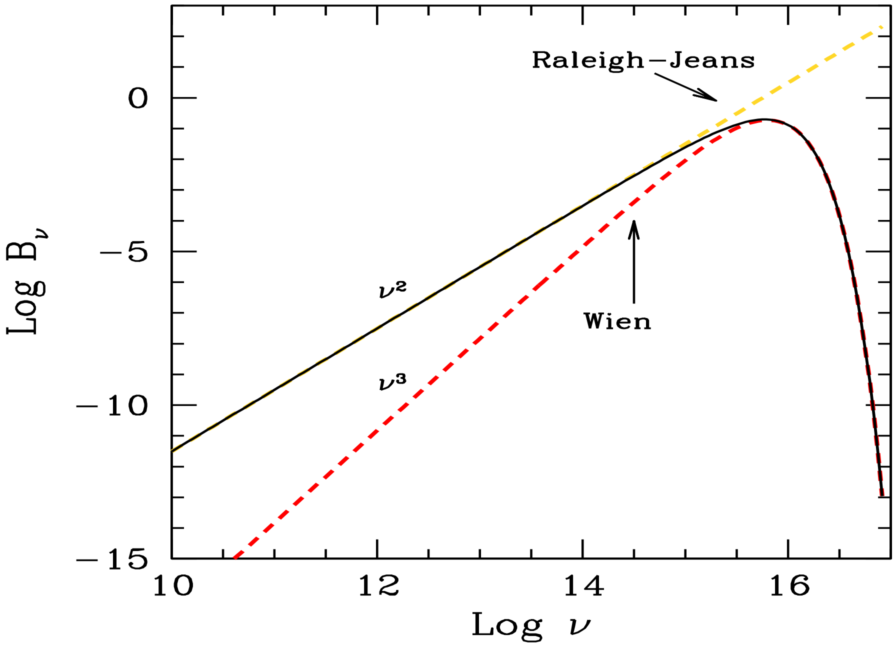

Figure 2.3: The black body intensity compared with the Raleigh–Jeans and the Wien law.

- The **brightness temperature** is defined using the Raleigh–Jeans law, since $I_\nu^{\rm RJ} = (2\nu^2/c^2)kT$ we have

$$T_{\rm b} = \frac{c^2 I_\nu^{\rm RJ}}{2k\nu^2} \tag{2.23}$$

- A black body is the most efficient radiator, for thermal plasmas and incoherent radiation (we can have coherent processes that are even more efficient). For a given surface and temperature, it is not possible to overtake the luminosity of the black body, at any frequency, for any emission process.

- Let us try to find the temperature of the surface of the Sun. We know its radius (700,000 km) and luminosity ($L_\odot = 4 \times 10^{33}$ erg s^{-1}). Therefore, from

$$L_\odot = \pi\, 4\pi R^2 \int_0^\infty B_\nu d\nu = 4\pi R^2 \sigma_{\rm MB} T^4 \tag{2.24}$$

we get:

$$T_\odot = \left(\frac{L_\odot}{4\pi R^2 \sigma_{\rm MB}}\right)^{1/4} \sim 5800\ K \tag{2.25}$$

- A spherical source emits black body radiation. We know its distance, but not its radius. Find it. Suppose we do not know its distance. Can we predict its angular size? And, if we then observe it, can we then get the distance?